Subsumptions of SPO Rules

Thierry Boy de la Tour
CNRS, Université Grenoble Alpes, France

TERMGRAPH 2024

Introduction

from: Subsumptions of Algebraic Rewrite Rules, ACT23

simultaneous application of rules, with merging of direct transformations through subsumptions

inspired by: Global Graph Transformations by Fernandez, Maignan, Spicher 2015-... and Parallel Coherent Graph Transformations by Boy de la Tour, Echahed, WADT 2020.

but: ACT23 not devoted to a single approach

Summary:

- partial transformations
- subsumptions
- tools for SPO transformations
- subsumptions of SPO rules
- final remarks

Algebraic approaches to GT

approach	features
DPO	existence of POC
SPO	removes unmatched items (non locality)
SqPO	duplications, non locality
PBPO	duplications, co-match, non locality

what do they have in common?

- they end in PO $\Rightarrow \exists$ a $\mathcal{C}$-span $D \xleftarrow{k} K \xrightarrow{r} R$
- D is obtained from G by deletion/duplication
  or: G is obtained from D by merging/extension
  or: $\exists f : D \to G$

Partial transformations

A partial transformation τ is a $\mathcal{C}$-diagram $G \stackrel{f}{\leftarrow} D \stackrel{k}{\leftarrow} K \stackrel{r}{\rightarrow} R$

approach = rule $\longleftarrow$ direct transformation $\longmapsto$ partial transformation

τ' subsumes τ if it can be obtained from τ by further deletions/duplications and further mergings/extensions

$\Rightarrow$ a subsumption morphism $\sigma : \tau \to \tau'$ is $(\sigma_1, \sigma_2, \sigma_3)$ such that

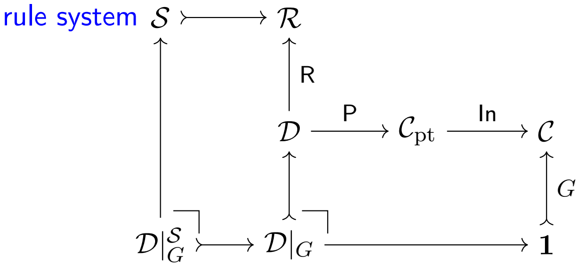
$$\begin{array}{ccccccc}
G & \stackrel{f}{\longleftarrow} & D & \stackrel{k}{\longleftarrow} & K & \stackrel{r}{\longrightarrow} & R \\
= \Big| & & \Big\uparrow \sigma_1 & & \Big\downarrow \sigma_2 & & \Big\downarrow \sigma_3 \\
G' & \underset{f'}{\longleftarrow} & D' & \underset{k'}{\longleftarrow} & K' & \underset{r'}{\longrightarrow} & R'
\end{array}$$

commutes $\quad\Rightarrow\quad$ category $\mathcal{C}_{\mathrm{pt}}$

Rewriting Environments (RE)

A Rewriting Environment for $\mathcal{C}$ is $\mathcal{R} \xleftarrow{\mathsf{R}} \mathcal{D} \xrightarrow{\mathsf{P}} \mathcal{C}_{\mathrm{pt}}$
where:

- $\mathcal{D}$ is a category of direct transformations and subsumptions
- $\mathcal{R}$ is a category of rules and subsumptions
- R and P are functors



The subsumptions in $\mathcal{D}|_G^{\mathcal{S}}$ are induced by the subsumptions in $\mathcal{S}$

Link from $\mathcal{R}$ to $\mathcal{D}$

problem: $\mathcal{D}$ discrete $\Rightarrow \mathcal{D}|_G^{\mathcal{S}}$ discrete

$$\longrightarrow \text{ in } \mathcal{R} \xleftarrow{\mathsf{R}} \mathcal{D} \xrightarrow{\mathsf{P}} \mathcal{C}_{\mathrm{pt}}$$ we need a constraint on $\mathcal{D}$-morphisms w.r.t. $\mathcal{R}$-morphisms

But R not full: take $\delta, \delta' \in \mathcal{D}$ s.t. $\mathsf{R}\,\delta = \mathsf{R}\,\delta' \ldots$

δ, δ' are related only if $\mathsf{R}\,\delta \xrightarrow{\sigma} \mathsf{R}\,\delta'$ and their matchings are related accordingly

problem: find largest possible $\mathcal{R}$ and $\mathcal{D}$ such that

$$\mathsf{R}\,\delta \xrightarrow{\sigma} \mathsf{R}\,\delta' \text{ and } m_\delta \xrightarrow{\sigma} m_{\delta'} \quad \Rightarrow \quad \exists! \mu : \delta \to \delta' \text{ s.t. } \mathsf{R}\,\mu = \sigma$$

in ACT23: done for DPO (in adhesive categories), SqPO and PBPO

SPO

Löwe, Algebraic Approach to Single-Pushout Graph Transformation, TCS93

graph structures $\rightarrow$ categories of presheaves $\widehat{\mathcal{C}} = \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ where $\mathcal{C}$ small

PO of partial morphisms are based on:

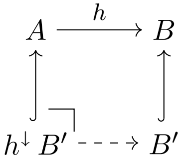
1. subcategory $\mathcal{I}$ of inclusion morphisms $A \overset{i}{\hookrightarrow} B$
2. direct images: for $A' \hookrightarrow A \xrightarrow{h} B$ there is a unique $h^{\uparrow} A'$ s.t.

$$\begin{array}{ccc} A & \xrightarrow{\;h\;} & B \\ \uparrow & & \uparrow \\ A' & \dashrightarrow\!\!\!\!\rightarrow & h^{\uparrow} A' \end{array}$$

commutes

SPO

3. inverse images: for $A \xrightarrow{h} B \hookleftarrow B'$ there is a unique $h^{\downarrow} B'$ s.t.



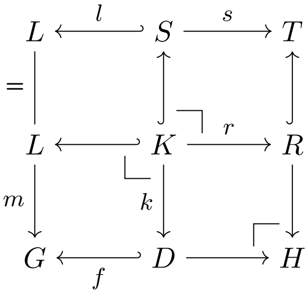
4. complete lattice $(\mathcal{I}, \sqcup, \sqcap)$ s.t.

$$h^{\uparrow}(A \sqcup B) = (h^{\uparrow} A) \sqcup (h^{\uparrow} B)$$

$$h^{\downarrow}(A \sqcup B) = (h^{\downarrow} A) \sqcup (h^{\downarrow} B)$$

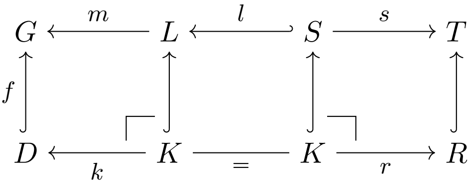
$$h^{\uparrow}(h^{\downarrow} B') = B' \sqcap (h^{\uparrow} A) \qquad \text{(for } A \xrightarrow{h} B \hookleftarrow B'\text{)}$$

SPO direct transformations



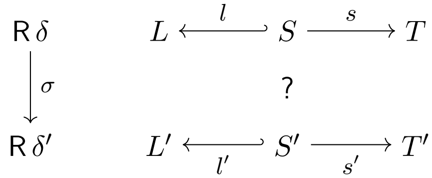
$$K = \sqcup\{J \hookrightarrow S \sqcap L \text{ s.t. } s^{\downarrow}(s^{\uparrow} J) = J \text{ and } m^{\downarrow}(m^{\uparrow} J) = J\}$$

$$R = \sqcup\{J \hookrightarrow T \text{ s.t. } s^{\downarrow} J \hookrightarrow K\} \text{ and } D = \sqcup\{J \hookrightarrow G \text{ s.t. } m^{\downarrow} J \hookrightarrow K\}$$

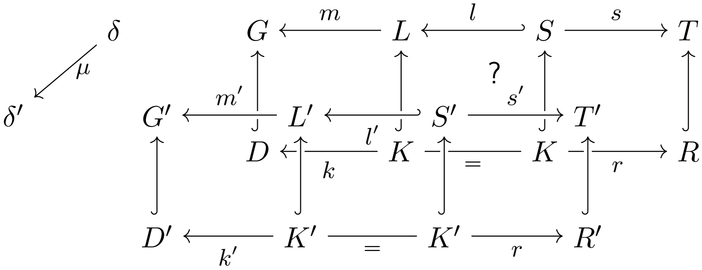


Subsumptions ?

if

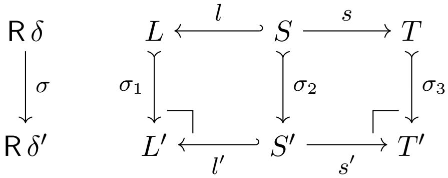


and $m \xrightarrow{\sigma} m'$ then $\exists! \mu : \delta \to \delta'$ s.t.

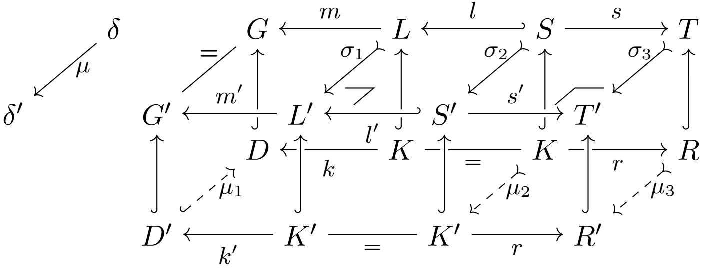


A theorem

if



and $m = m' \circ \sigma_1$ then $\exists! \mu = (\mu_1, \mu_2, \mu_3, \sigma_1, \sigma_2, \sigma_3)$ s.t.



A strange lemma

Lemma

If
$$\begin{array}{ccccc} J & \hookrightarrow & B & \xleftarrow{\;h\;} & A \\ & & {\scriptstyle k}\downarrow & \lrcorner & \downarrow \\ & & D & \longleftarrow & C \end{array}$$
and $k^{\downarrow}(k^{\uparrow}(J \sqcap h^{\uparrow} A)) = J \sqcap h^{\uparrow} A$
then $k^{\downarrow}(k^{\uparrow} J) = J$

Proof.

if not then $\exists s \in \mathcal{C}$, $\exists x \in J_s$ s.t. $k_s^{\downarrow}(k_s(x)) \nsubseteq J_s$
hence $\exists x' \in B_s \backslash J_s$ s.t. $k_s(x') = k_s(x)$

$$\begin{array}{ccccc} x \neq x' \in B_s & \xleftarrow{\;h_s\;} & A_s \\ {\scriptstyle k_s}\downarrow & \lrcorner & \downarrow \\ k_s(x) \in D_s & \longleftarrow & C_s \end{array}$$

by GC $x', x \in h_s^{\uparrow} A_s$, hence $x \in (J \sqcap h^{\uparrow} A)_s$
hence $x' \in k_s^{\downarrow}(k_s(x)) \subseteq (J \sqcap h^{\uparrow} A)_s$ contradiction

Comparison with DPO

$$
\begin{array}{ccccc}
L & \xleftarrow{\;l\;} & K & \xrightarrow{\;r\;} & R \\
\downarrow{\scriptstyle \sigma_1} & & \downarrow{\scriptstyle \sigma_2} & & \downarrow{\scriptstyle \sigma_3} \\
L' & \xleftarrow[\;l'\;]{} & K' & \xrightarrow[\;r'\;]{} & R'
\end{array}
$$

with $\widehat{\mathcal{C}}$, $\mathcal{R}_{\mathrm{SPO}}$ is a subcategory of $\mathcal{R}_{\mathrm{DPO}}$

$$
\begin{array}{ccccc}
\mathcal{R}_{\mathrm{DPO}} & \xleftarrow{\;\mathsf{R}_{\mathrm{DPO}}\;} & \mathcal{D}_{\mathrm{DPO}} & \xrightarrow{\;\mathsf{P}_{\mathrm{DPO}}\;} & \mathcal{C}_{\mathrm{pt}} \\
\uparrow & & \uparrow & & \| \\
\mathcal{R}_{\mathrm{SPO}} & \longleftarrow & \mathcal{D}^{\hookrightarrow}_{\mathrm{DPO}} & & = \\
= \, \| & & \downarrow & & \| \\
\mathcal{R}_{\mathrm{SPO}} & \xleftarrow[\;\mathsf{R}_{\mathrm{SPO}}\;]{} & \mathcal{D}_{\mathrm{SPO}} & \xrightarrow[\;\mathsf{P}_{\mathrm{SPO}}\;]{} & \mathcal{C}_{\mathrm{pt}}
\end{array}
$$