

ZX-calculus as a graphical rewriting language for quantum computing

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ZX-calculus

Spiders

$$\begin{array}{c} m \\ \vdots \\ \text{---} \\ \alpha \\ \text{---} \\ \vdots \\ n \end{array} \quad \xrightarrow{\llbracket \cdot \rrbracket} \quad |0\rangle^{\otimes n} \langle 0|^{\otimes m} + e^{i\alpha} |1\rangle^{\otimes n} \langle 1|^{\otimes m}$$

$$\begin{array}{c} m \\ \vdots \\ \text{---} \\ \alpha \\ \text{---} \\ \vdots \\ n \end{array} \quad \xrightarrow{\llbracket \cdot \rrbracket} \quad |+\rangle^{\otimes n} \langle +|^{\otimes m} + e^{i\alpha} |-\rangle^{\otimes n} \langle -|^{\otimes m}$$

Computational basis states

$$\text{---} \xrightarrow{\llbracket \cdot \rrbracket} |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

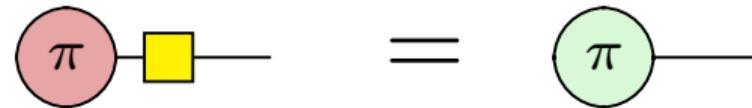
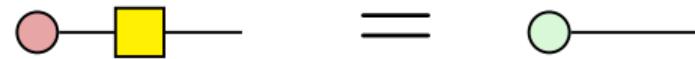
$$\pi \text{---} \xrightarrow{\llbracket \cdot \rrbracket} |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Superposition

$$\text{---} \quad \xrightarrow{\mathbb{E}} \quad |+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\pi \text{---} \quad \xrightarrow{\mathbb{E}} \quad |-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Hadamard box



Hadamard box

$$\text{Red circle} \otimes \text{Yellow square} = \text{Green circle}$$

$$\text{Red circle with } \pi \otimes \text{Yellow square} = \text{Green circle with } \pi$$

$$\text{Yellow square} \mapsto |\cdot\rangle\langle 0| + |\cdot\rangle\langle 1|$$

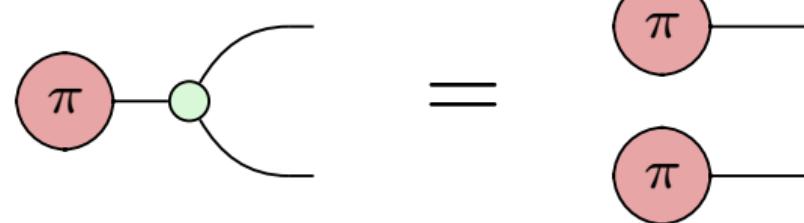
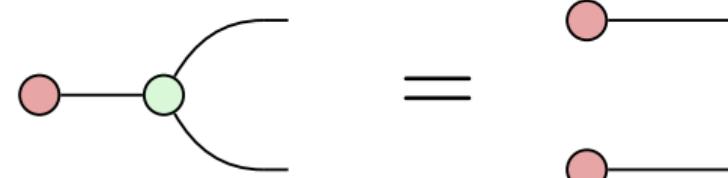
Hadamard matrix

$$\text{---} \boxed{} \text{---} \quad \mapsto \quad |+\rangle \langle 0| + |-\rangle \langle 1|$$

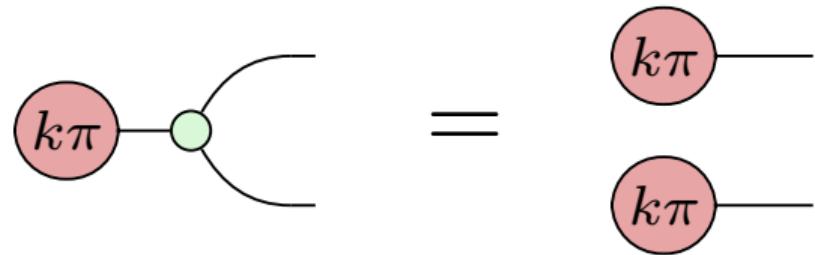
$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

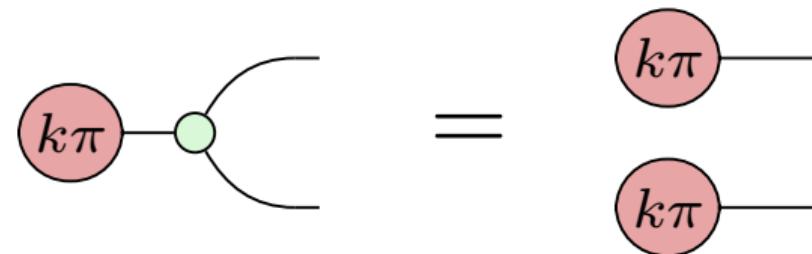
Z-spider



Z-spider



Z-spider



A diagram illustrating the action of a Z-spider on a state. On the left, a horizontal line enters a green circle, which then splits into two curved lines that meet at a point below the green circle. This is followed by a mapping arrow \mapsto with a bracketed expression $[\cdot]$ above it. To the right of the arrow, the expression $|0\rangle \langle 0,0| + |1\rangle \langle 1,1|$ is shown.

X-spider

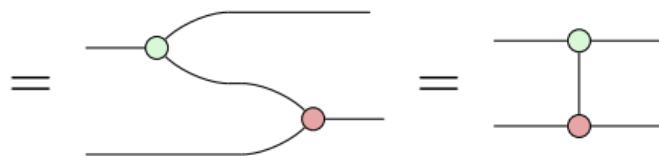
$$\begin{array}{c} j\pi \\ \text{---} \\ k\pi \end{array} = \text{---} \quad (j \oplus k)\pi$$

The diagram illustrates a commutative property of a specific operation. On the left, two circular nodes are shown: one at the top labeled $j\pi$ and one at the bottom labeled $k\pi$. They are connected by a curved line that merges into a single horizontal line that extends to the right. This is followed by an equals sign. To the right of the equals sign is a single horizontal line labeled $(j \oplus k)\pi$, enclosed in a rounded rectangular box. This visualizes the fact that the combined effect of $j\pi$ and $k\pi$ is equivalent to a single operation labeled $(j \oplus k)\pi$.

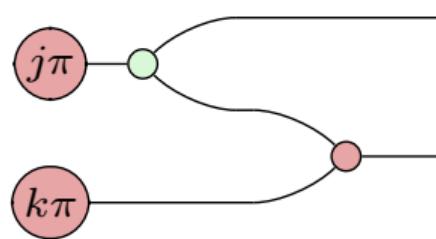
Composition

$$[(\text{---}) \otimes (\text{---} \circ \text{---})] \circ [(\text{---} \circ \text{---}) \otimes (\text{---})]$$

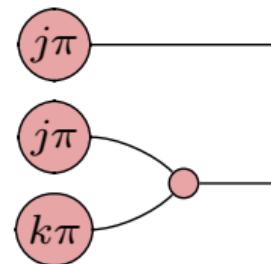
$$= \left[\begin{array}{c} \text{---} \\ \text{---} \circ \text{---} \end{array} \right] \circ \left[\begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \end{array} \right]$$

$$= \text{---} \circ \text{---} = \text{---} \circ \text{---}$$


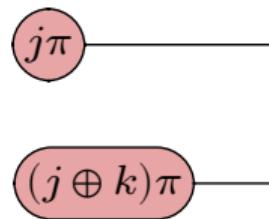
CNOT gate



=



=



Phases

$$\text{---} \circlearrowleft \alpha \text{---} = |0\rangle\langle 0| + e^{i\alpha} |1\rangle\langle 1|$$

$$\text{---} \circlearrowleft \alpha \text{---} = |+\rangle\langle +| + e^{i\alpha} |-\rangle\langle -|$$

Phases

$$\text{---} \circlearrowleft \alpha \circlearrowright \text{---} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{bmatrix}$$

$$\text{---} \circlearrowleft \alpha \circlearrowright \text{---} = \frac{1}{2} \begin{bmatrix} 1 + e^{i\alpha} & 1 - e^{i\alpha} \\ 1 - e^{i\alpha} & e^{i\alpha} \end{bmatrix}$$

Quantum gates

$$\text{NOT} = \begin{array}{c} \text{---} \\ \oplus \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array}$$

$$= \begin{array}{c} \text{---} \\ \pi \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array}$$

$$\text{CNOT} = \begin{array}{c} \text{---} \\ \bullet \\ \text{---} \\ \oplus \end{array} = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array}$$

$$= \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ -\frac{\pi}{2} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array}$$

$$\text{HAD} = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \square \end{array}$$

$$= \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \frac{\pi}{4} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array}$$

$$= \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ -\frac{\pi}{4} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array}$$

Spiders

$$\begin{array}{c} m \\ \vdots \\ \text{---} \\ \alpha \\ \text{---} \\ \vdots \\ n \end{array} \quad \xrightarrow{\llbracket \cdot \rrbracket} \quad |0\rangle^{\otimes n} \langle 0|^{\otimes m} + e^{i\alpha} |1\rangle^{\otimes n} \langle 1|^{\otimes m}$$

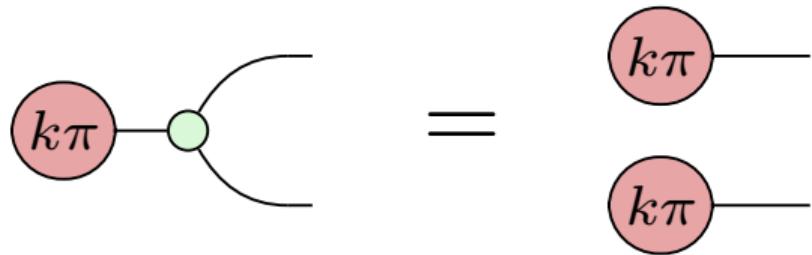
$$\begin{array}{c} m \\ \vdots \\ \text{---} \\ \alpha \\ \text{---} \\ \vdots \\ n \end{array} \quad \xrightarrow{\llbracket \cdot \rrbracket} \quad |+\rangle^{\otimes n} \langle +|^{\otimes m} + e^{i\alpha} |-\rangle^{\otimes n} \langle -|^{\otimes m}$$

Theorem (Universality)

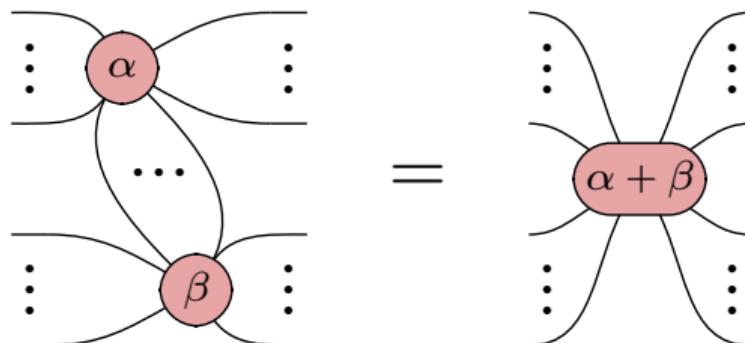
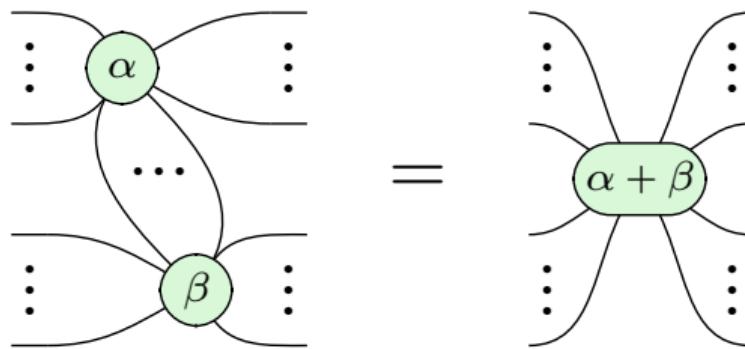
Any linear map between qubits can be expressed in terms of ZX diagrams.

Rewrite rules

Copy



Fusion

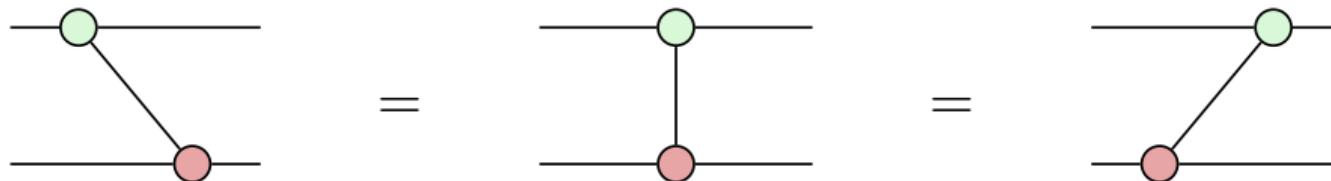


Color

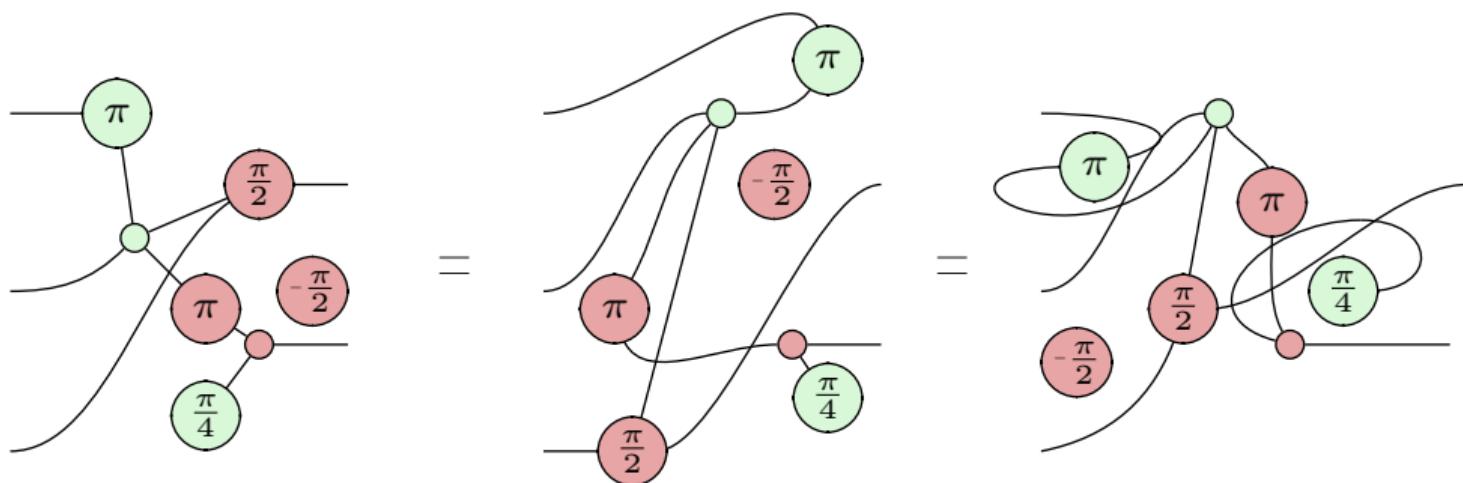
$$\begin{array}{c} \text{Diagram 1: } \\ \text{Left: A green circle labeled } \alpha \text{ with four yellow square inputs and two curved output lines connecting to the top and bottom.} \\ \text{Right: An equals sign followed by a red circle labeled } \alpha \text{ with two curved output lines connecting to the top and bottom.} \end{array}$$

$$\begin{array}{c} \text{Diagram 2: } \\ \text{Left: A red circle labeled } \alpha \text{ with four yellow square inputs and two curved output lines connecting to the top and bottom.} \\ \text{Right: An equals sign followed by a green circle labeled } \alpha \text{ with two curved output lines connecting to the top and bottom.} \end{array}$$

Only Connectivity Matters



Only Connectivity Matters



Axioms

$$\begin{array}{ccc} \text{Diagram showing two green circles labeled } \alpha \text{ and } \beta \text{ with multiple outgoing lines, connected by a horizontal line with an equals sign.} & & \\ (\text{FUSION}) & = & \text{Diagram showing a single green circle labeled } \alpha + \beta \text{ with multiple outgoing lines.} \end{array}$$

$$\begin{array}{ccc} \text{Diagram showing a green dot on a line, followed by an equals sign.} & & \text{Diagram showing a red dot on a line, followed by an equals sign.} \\ (\text{Z-ELIM}) & = & (\text{X-ELIM}) \\ \text{Diagram showing a yellow square on a line, followed by an equals sign.} & & \text{Diagram showing three circles: green } \frac{\pi}{2}, red dot, green } \frac{\pi}{2}, followed by an equals sign. \\ (\text{EULER}) & = & \end{array}$$

$$\begin{array}{ccc} \text{Diagram showing a green circle labeled } \alpha \text{ with two yellow squares on its left, connected by a horizontal line with an equals sign.} & & \\ (\text{COLOUR}) & = & \text{Diagram showing a red circle labeled } \alpha \text{ with multiple outgoing lines.} \end{array}$$

$$\begin{array}{ccc} \text{Diagram showing three red circles labeled } \alpha_1, \alpha_2, \alpha_3 \text{ connected by a horizontal line, followed by an equals sign.} & & \text{Diagram showing three circles: red } \alpha, green } \pi, green } \gamma, followed by an equals sign. \\ (\ast) & = & \text{Diagram showing three green circles labeled } \beta_1, \beta_2, \beta_3 \text{ connected by a horizontal line.} \end{array}$$

$$\begin{array}{ccc} \text{Diagram showing a red dot and a green dot connected by a line, followed by an equals sign.} & & \text{Diagram showing a red dot on a line.} \\ (\text{COPY}) & = & \end{array}$$

$$\begin{array}{ccc} \text{Diagram showing a red dot and a green dot connected by a line, followed by an equals sign.} & & \text{Diagram showing a green circle labeled } \frac{\pi}{4} \text{ and a red circle labeled } -\frac{\pi}{4} \text{ connected by a line, followed by an equals sign.} \\ (\text{BIGEBRA}) & = & \text{Diagram showing a red dot and a green dot connected by a line.} \\ \text{Diagram showing a red dot and a green dot connected by a line, crossed by another line, followed by an equals sign.} & & \end{array}$$

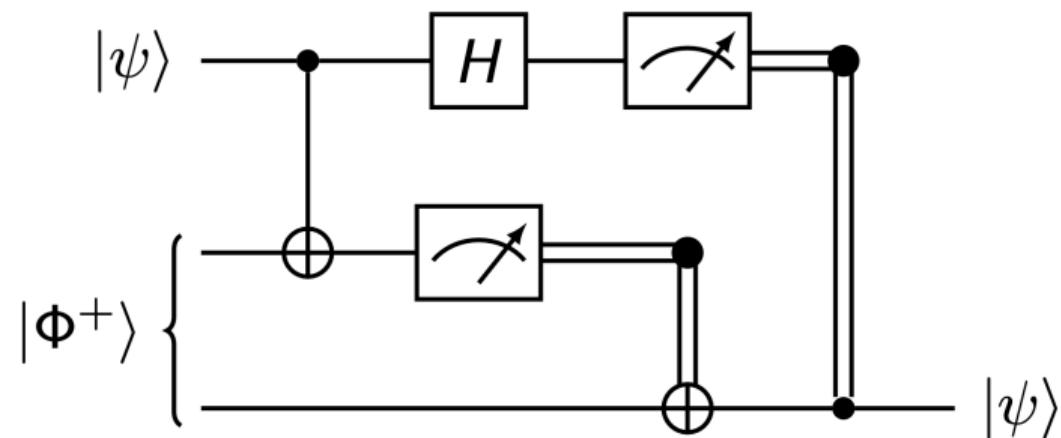
Completeness

Theorem (Completeness)

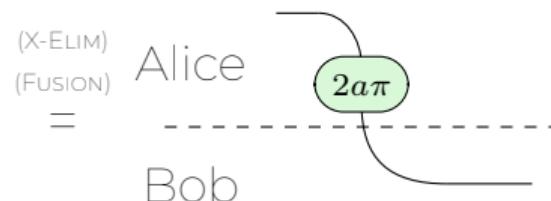
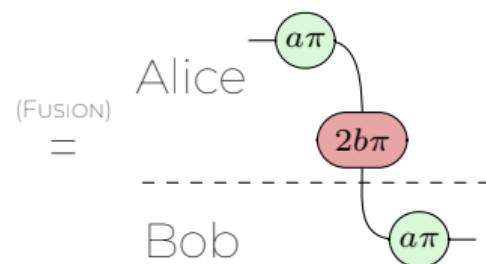
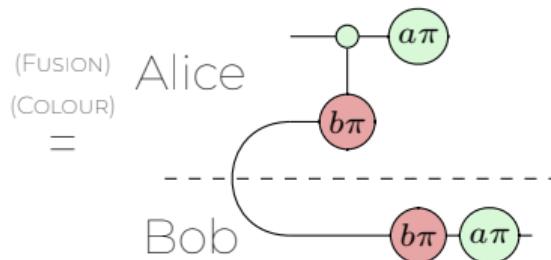
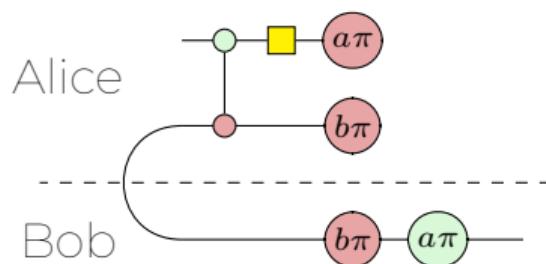
Any equation that holds for linear maps between qubits can be derived in ZX-calculus.

Examples

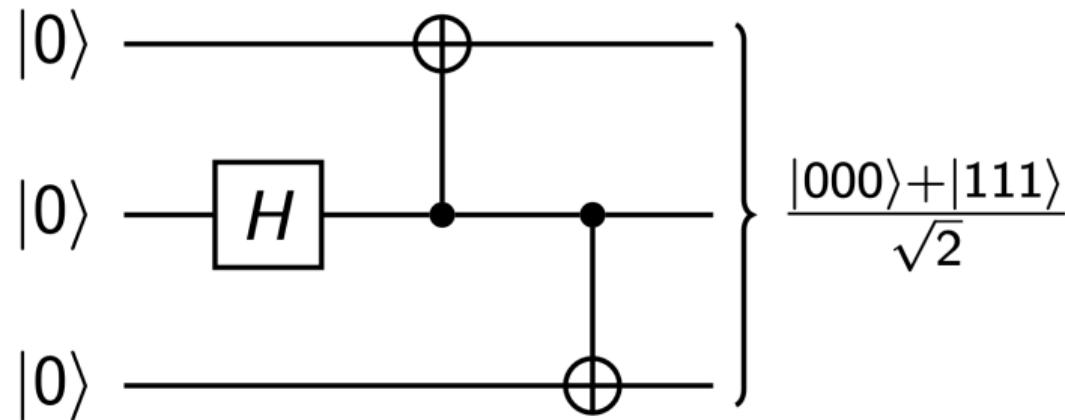
Quantum Teleportation



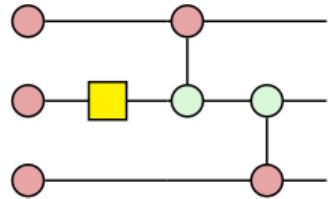
Quantum Teleportation



GHZ state preparation

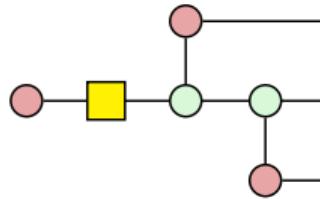


GHZ state preparation



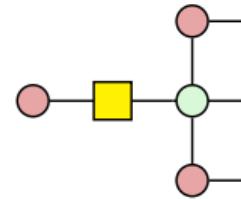
(FUSION)

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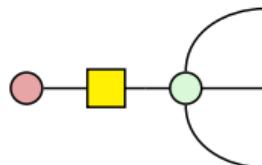
(FUSION)

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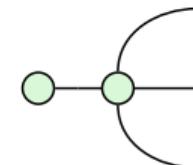
(X-ELIM)

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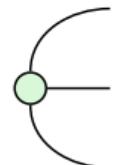
(COLOUR)

=



(FUSION)

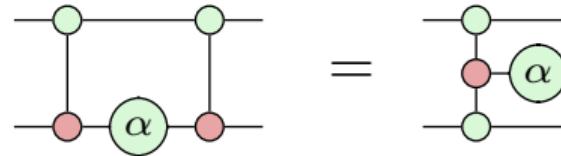
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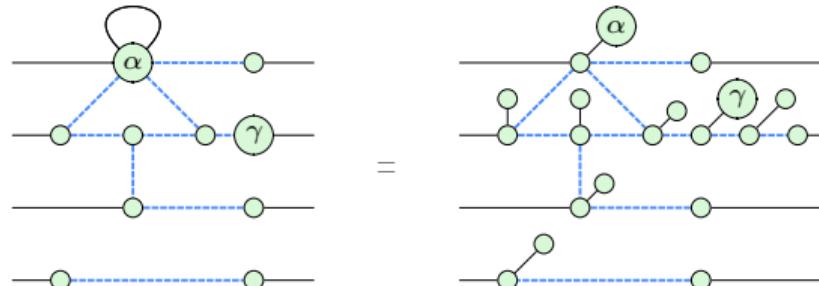
Extensions

Applications: ZX-calculus

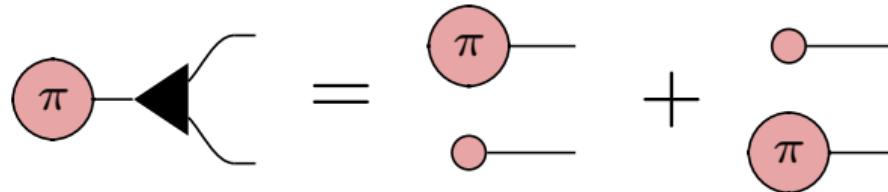
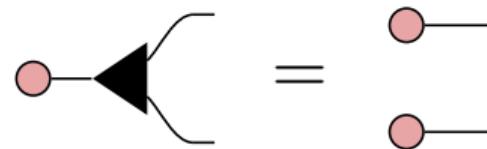
Quantum Circuit Optimisation



Measurement-Based Quantum Computing



W node



ZW-calculus

Summation

$$\begin{array}{c} \text{---} \nearrow \square \xrightarrow{\quad a \quad} \searrow \text{---} \\ \text{---} \nearrow \square \xrightarrow{\quad b \quad} \searrow \text{---} \end{array} = \begin{array}{c} \text{---} \xrightarrow{\quad a + b \quad} \text{---} \end{array}$$

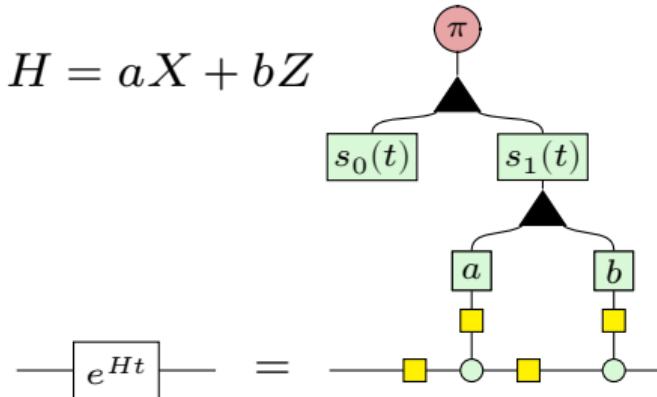
Linear Optical Quantum Computing

$$\begin{array}{c} \text{---} \nearrow \square \xrightarrow{\quad a \quad} \searrow \text{---} \\ \text{---} \nearrow \square \xrightarrow{\quad b \quad} \searrow \text{---} \end{array} = \begin{array}{c} \text{---} \nearrow \square \xrightarrow{\quad a + b \quad} \searrow \text{---} \\ \text{---} \nearrow \square \xrightarrow{\quad b \quad} \searrow \text{---} \end{array}$$

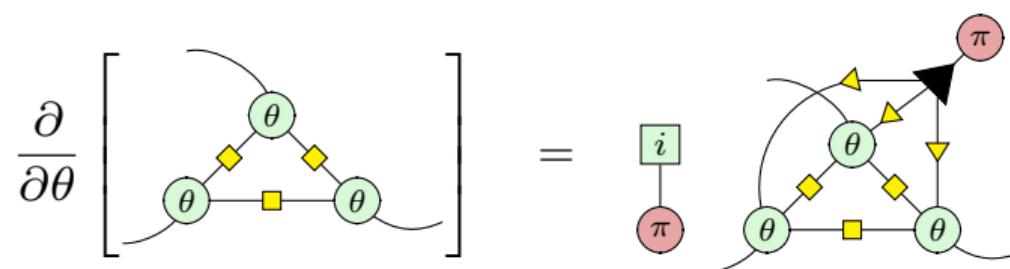
ZXW-calculus

Hamiltonians

$$H = aX + bZ$$



Differentiation and integration



Thank you!

Overview

1 Introduction

2 Generators

3 Rewrite rules

4 Examples

5 Extensions